Classification of Möbius Maps

Fixed Points, Elliptic Maps, and Apollonius–Steiner Circles

Lecture Notes

October 20, 2026

1 Fixed Points and the Trace Criterion

Let T be a Möbius map, $T \neq \mathrm{id}$. We look for its **fixed points** (possibly ∞):

$$z = \frac{az+b}{cz+d} \iff cz^2 + (d-a)z - b = 0.$$

Normalize so that $ad - bc = 1$.

Case count

Case 1 (two fixed points): $(d-a)^2 \neq -4bc \iff (a+d)^2 \neq 4$.

Case 2 (one fixed point, "parabolic"): $(d-a)^2 = -4bc \iff a + d = \mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 2$.

2 Case 1: Two Fixed Points

Let z_1, z_2 be the fixed points; start with the normalized case $z_1 = 0$, $z_2 = \infty$. Then $c = 0$, $b = 0$, $ad = 1$, and

$$Tz = \frac{a}{d} z = a^2 z.$$

Case 1(a): $|a^2| = 1$, $a^2 \neq 1$ – elliptic. Write $a^2 = e^{i\theta}$; then $Tz = e^{i\theta} z$ is a rotation. It rotates the rays $\{\arg z = \varphi\}$ to $\{\arg z = \varphi + \theta\}$, and leaves each circle $C_r = \{|z| = r\}$ invariant.

$$\mathrm{Tr}\, T = a + \frac{1}{a} = e^{i\theta/2} + e^{-i\theta/2} = 2\cos\frac{\theta}{2} \in (-2, 2), \qquad \theta \in (0, 2\pi).$$

Arbitrary fixed points z_1, z_2. Let $Sz := \dfrac{z - z_1}{z - z_2}$, so $Sz_1 = 0$, $Sz_2 = \infty$, and set $\widetilde{T} := STS^{-1}$. Then

$$\widetilde{T}(\infty) = ST(z_2) = S(z_2) = \infty, \qquad \widetilde{T}(0) = ST(z_1) = S(z_1) = 0,$$

so $\widetilde{T}$ is exactly T in normalized position, and $\mathrm{Tr}\, \widetilde{T} = \mathrm{Tr}\, T$ (conjugate matrices have equal trace).

Definition (elliptic map)

T is **elliptic** if $\widetilde{T} = STS^{-1}$ is elliptic. Equivalently, $\mathrm{Tr}\, T \in (-2, 2)$. In this case $T = S^{-1}\widetilde{T}S$.

3 Geometry: Apollonius and Steiner Circles

Definition

$$C_r(z_1, z_2) := \{|Sz| = r\} = \left\{\frac{|z - z_1|}{|z - z_2|} = r\right\} \quad \text{– circles of Apollonius,}$$

$$L_\varphi(z_1, z_2) := \{\arg(Sz) = \varphi\} \quad \text{– circular arcs from } z_1 \text{ to } z_2 \text{ (Steiner circles).}$$

Since S preserves angles, $L_\varphi(z_1, z_2) \perp C_r(z_1, z_2)$ for every φ, r: the Apollonius circles and the Steiner arcs form two mutually orthogonal families, both passing through (or separating) z_1, z_2.

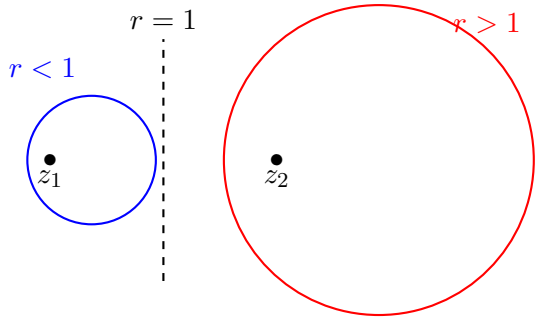


Figure 1: The circles of Apollonius $C_r(z_1, z_2)$, recreated in TIkZ from the original hand-drawn sketch: as r ranges over $(0, \infty)$, the circles shrink toward z_1 (as $r \to 0$) or z_2 (as $r \to \infty$), and degenerate to the perpendicular bisector of z_1, z_2 exactly when $r = 1$.

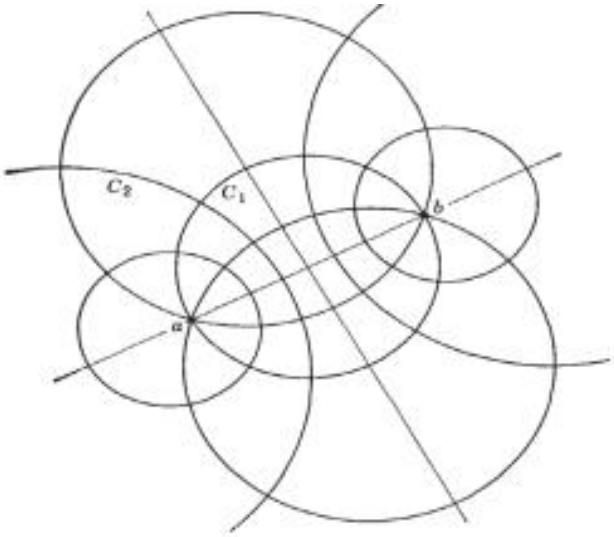


Figure 2: The original illustration of the two mutually orthogonal families: Apollonius circles $C_r(z_1, z_2)$ and Steiner arcs $L_\varphi(z_1, z_2)$, through the points $a = z_1$, $b = z_2$.

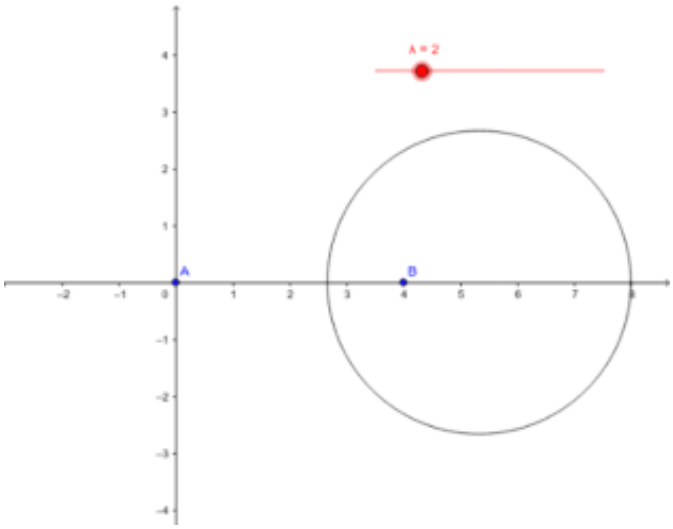


Figure 3: The original interactive GeoGebra illustration: a single Apollonius circle for z_1, z_2 marked on the real axis, with a slider controlling the ratio r (the "dynamics" – dragging the slider varies r and the circle updates live). Live applet: `https://www.geogebra.org/m/sThyBeRS`.